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# A New Semifield Flock*

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We construct the new semifield flock of $PG(3, 243)$ associated with the Penttila–Williams translation ovoid of $Q(4, 243)$ and we study the associated generalized quadrangle and its translation dual. © 1999 Academic Press

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## 1. INTRODUCTION

Let $K$ be a quadratic cone of $PG(3, q)$ with vertex $v$. A *flock* $\mathscr{F}$ of $K$ is a partition of $K\backslash\{v\}$ into $q$ conics. If all planes containing the elements of the flock $\mathscr{F}$ share a common line, then $\mathscr{F}$ is called *linear*. Given a flock $\mathscr{F}$ there is a standard construction of a generalized quadrangle $Q(\mathscr{F})$ associated with $\mathscr{F}$ ([19]), which is classical if and only if $\mathscr{F}$ is a linear flock.

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Denote by $S(\mathscr{F})$ the spread of $PG(3, q)$ associated with $\mathscr{F}$ ([4, 25, or 5]). When the translation plane constructed using $S(\mathscr{F})$ is a semifield plane, $\mathscr{F}$ is called a *semifield flock*. When $q$ is even, N. L. Johnson [7] has shown that all semifield flocks are linear. As we are interested in non-linear semifield flocks, we **always** suppose $q$ is an odd prime power.

Let $q = p^e$, $p$ any odd prime, and $K$ be the cone with equation $x_0 x_1 = x_2^2$ and vertex $(0, 0, 0, 1)$. Two classes of semifield flocks are known.

The $q$ planes $\pi_t$ with equation $tx_0 - mt^\sigma x_1 + x_3 = 0$, $t \in GF(q)$, $m$ a given non-square of $GF(q)$, and $\sigma$ a given automorphism of $GF(q)$, define a semifield flock of $K$ ([5] and [19]). All the planes $\pi_t$ contain the point $(0, 0, 1, 0)$. This flock is linear if and only if $\sigma = 1$. Conversely, every non-linear flock of $K$ for which the planes of the $q$ conics all contain a common point, is of the type just described ([19]). The flock is called the *Kantor semifield flock.*

Let $q = 3^r$ and $r > 2$. Then the $q$ planes $\pi_t$ with equation $tx_0 - (m^{-1}t^9 + mt)$ $x_1 - t^3 x_2 + x_3 = 0$, $t \in GF(q)$, $m$ a given non-square of $GF(q)$, define a semifield flock of $K$ (see [5] where the description of the flock is different from the one above which is taken from [14]). This flock is called the *Ganley flock*.

Let $\mathscr{F}$ be a flock of $K$, i.e., $\mathscr{F} = \{K \cap \pi_t \mid t \in GF(q)\}$ and, following [5], $\pi_t$ is the plane with equation $tx_0 - f(t)\, x_1 + g(t)\, x_2 + x_3 = 0$, where $f$ and $g$ are maps from $GF(q)$ to itself. One can suppose $f(0) = g(0) = 0$. We write $\mathscr{F} = \mathscr{F}(f, g)$ and we say that $\pi_t$ is a plane of the flock. The flock $\mathscr{F}(f, g)$ is a semifield flock if and only if $f$ and $g$ are additive maps ([5] (6.5)).

L. Bader and G. Lunardon ([2]) have proved that if $\mathscr{F}(f, g)$ is a semifield flock and there is a polynomial $h(t)$ over $GF(q)$ such that for a fixed non-square $m$ in $GF(q)$ the equation $g^2(t) + 4tf(t) = mh^2(t)$ is a polynomial identity, then $\mathscr{F}(f, g)$ is one of the known examples.

We observe that if $g^2(t) + 4tf(t) = mh^2(t)$ is a polynomial identity, for any odd integer $r$ the polynomials $f$, $g$, and $h$ define functions $f_r$, $g_r$, and $h_r$ respectively from $GF(q^r)$ to itself such that $g_r^2(t) + 4tf_r(t) = mh_r^2(t)$ for all $t \in GF(q^r)$. Therefore, $\mathscr{F}(f_r, g_r)$ is a semifield flock of the quadratic cone $x_2^2 - x_0 x_1 = 0$ of $PG(3, q^r)$ containing $\mathscr{F}(f, g)$. Hence, if $g^2(t) + 4tf(t) = mh^2(t)$ is not a polynomial identity, then the semifield flock $\mathscr{F}(f, g)$ is *sporadic.*

The point-line dual of $Q(\mathscr{F})$ is a translation generalized quadrangle if and only if $\mathscr{F}$ is a semifield flock ([7]), i.e., there is an egg $\mathscr{E}_{\mathscr{F}}$ such that $T(\mathscr{E}_{\mathscr{F}})$ is the point-line dual of $Q(\mathscr{F})$. Moreover its translation dual is defined by a good egg $\mathscr{E}^*_{\mathscr{F}}$ of $\Sigma = PG(4n-1, s)$ with $q = s^n$ ([20] and [23]).

J. A. Thas ([23] Main Theorem) proved that if $\Sigma$ is regarded as a canonical subgeometry of $\Sigma^* = PG(4n-1, s^r)$, there is a subspace $U$ of $\Sigma^*$

which meets in a point all the elements of $\mathscr{E}^*_{\mathscr{F}}$ and which satisfies one of the following conditions:

(a) $U$ has dimension 3 and $\mathscr{V} = \{x \in U \mid \exists X \in \mathscr{E}^*_{\mathscr{F}} : x \in X\}$ is an elliptic quadric of $U$, or

(b) $U$ has dimension 4 and $\mathscr{V} = \{x \in U \mid \exists X \in \mathscr{E}^*_{\mathscr{F}} : x \in X\}$ is the projection of a Veronese surface from a point, or

(c) $U$ has dimension 5 and $\mathscr{V} = \{x \in U \mid \exists X \in \mathscr{E}^*_{\mathscr{F}} : x \in X\}$ is a Veronese surface.

In [23] it is shown that condition (a) implies that $\mathscr{F}$ is a linear flock and (b) implies that $\mathscr{F}$ is a Kantor flock. Therefore, if $\mathscr{F}$ is the Ganley flock, then $\mathscr{E}^*_{\mathscr{F}}$ satisfies (c).

Because of these strong results it has been conjectured in [23] that if a good egg $\mathscr{E}^*_{\mathscr{F}}$ satisfies (c), then $\mathscr{F}$ must be a Ganley flock. This is equivalent to saying that there are no sporadic semifield flocks. But T. Penttila and B. Williams [18] have constructed a new translation ovoid of $Q(4, 243)$ with the aid of a computer, and by [12] (see also [3]) this new ovoid corresponds to a sporadic semifield flock of $PG(3, 243)$.

In this paper we give an explicit construction of this sporadic flock giving the equations of its planes. Also, we investigate the associated translation generalized quadrangle, proving that it is not isomorphic to any of the known examples, and we study its translation dual showing that it is neither the point-line dual of a flock quadrangle nor isomorphic to the Payne's Roman generalized quadrangle.

We express our thanks to J. A. Thas for having pointed out a gap in the proof of Theorem 1 in an earlier version of this paper.

## 2. TRANSLATION QUADRANGLES

Let $\mathscr{F} = \mathscr{F}(f, g)$ be a flock of the quadratic cone $K$ of $PG(3, q)$. Combining results of W. M. Kantor [9], S. E. Payne [13] and J. A. Thas [19], it is now a famous construction that a generalized quadrangle $Q(\mathscr{F})$ of order $(q^2, q)$ can be obtained as a coset geometry from a flock $\mathscr{F}$ of the quadratic cone via $q$-clans. (For more details see [16], Chapter 8.) The generalized quadrangle $Q(\mathscr{F})$ is classical if and only if $\mathscr{F}$ is linear [19].

N. L. Johnson ([7]) has shown that $\mathscr{F}$ is a semifield flock if and only if $Q(\mathscr{F})$ is a translation generalized quadrangle with base line $[A(\infty)]$.

Furthermore, we recall the following construction. An *egg*, $\mathscr{E}$, is a partial spread of $(n-1)$-dimensional subspaces of $PG(2n+m-1, s)$ such that:

(1) $\mathscr{E}$ contains $s^m+1$ elements;

(2) every three elements of $\mathscr{E}$ generate a $(3n-1)$-dimensional subspace of $PG(2n+m-1, s)$;

(3) each element $X$ of $\mathscr{E}$ is contained in a $(n+m-1)$-dimensional subspace $T_X$ having no point in common with any element of $\mathscr{E}$ different from $X$. The subspace $T_X$ is called the *tangent space* of $\mathscr{E}$ to $X$.

Embed $PG(2n+m-1, s)$ in $PG(2n+m, s)$ as a hyperplane, and define an incidence structure $T(\mathscr{E})$ as follows. Points are (i) the points of $PG(2n+m, s)\backslash PG(2n+m-1, s)$, (ii) the $(n+m)$-dimensional subspaces $X$ of $PG(2n+m, s)$ for which $X\cap PG(2n+m-1, s)$ is an element of $\mathscr{E}$, and (iii) a new symbol $(\infty)$. Lines are (a) the $n$-dimensional subspaces of $PG(2n+m, s)$ which are not contained in $PG(2n+m-1, s)$ and meet $PG(2n+m-1, s)$ in an element of $\mathscr{E}$, and (b) the elements of $\mathscr{E}$. Incidence is defined as follows: A point of type (i) is incident only with lines of type (a); here the incidence is that of $PG(2n+m, s)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathscr{E}$ incident with it. The point $(\infty)$ is incident with no line of type (a) and all lines of type (b).

S. E. Payne and J. A. Thas ([16] 8.7.1) proved that $T(\mathscr{E})$ is a translation generalized quadrangle of order $(s^n, s^m)$ with base point $(\infty)$, and conversely every translation generalized quadrangle, $\mathscr{S}$, of order $(r, t)$, with base point $x$ and translation group $T$, is isomorphic to some $T(\mathscr{E})$ in the following way: Fix a point $y$ of $\mathscr{S}$ not collinear with $x$, and let $L_0, L_1, ..., L_t$ be the lines of $\mathscr{S}$ incident with $x$. For $i=0, 1, ..., t$ let $z_i$ be the point of $L_i$ collinear with $y$, and denote by $M_i$ the line of $\mathscr{S}$ joining $y$ and $z_i$. Define $T_i=\{\tau\in T \mid M_i\tau=M_i\}$, $T_i^*=\{\tau\in T \mid z_i\tau=\tau\}$, $\mathscr{E}=\{T_i \mid i=0, 1, ..., t\}$. Then $T_i$ and $T_i^*$ are subgroups of $T$ such that $T_i$ has order $t$, $T_i^*$ has order $rt$ and $T_i<T_i^*$. The *kernel* $K$ of $\mathscr{S}$ is the set of all endomorphisms $\alpha$ of $T$ such that $T_i\alpha\subset T_i$, $0\leqslant i\leqslant t$. By [16] (8.5.1), $K$ is a field, $T$ is a vector space over the subfield $GF(s)$ of $K$, and $T_i$ and $T_i^*$ are $GF(s)$-vector subspaces for all $i=0, 1, ..., t$. As $T$ has cardinality $r^2t$, if $r=s^n$ and $t=s^m$, then $\mathscr{E}$ is an egg of $PG(2n+m-1, s)=PG(t, GF(s))$ and $\mathscr{S}$ is canonically isomorphic to $T(\mathscr{E})$. As $T$ is transitive on the points not collinear with $x$, the construction of $\mathscr{E}$ does not depend on $y$.

Let $\mathscr{S}'$ be another translation generalized quadrangle with base point $x'$ and denote by $T'$ the translation group of $\mathscr{S}'$. Denote by $\mathscr{E}'$ the egg of $T'$ constructed starting from the point $y'$ so that $\mathscr{S}'$ is isomorphic to $T(\mathscr{E}')$, and let $K'$ be the kernel of $\mathscr{S}'$. Suppose that $\sigma$ is an isomorphism of $\mathscr{S}$ into $\mathscr{S}'$ which maps $x$ to $x'$ and $y$ to $y'$. As the translation group of a finite translation generalized quadrangle $\mathscr{S}$ is uniquely defined ([16] 8.3.2), the map $\sigma^{-1}\tau\sigma$ is a translation of $\mathscr{S}'$ with base point $x'$, and the map $\alpha$ from

$T$ to $T'$ defined by $\tau \mapsto \sigma^{-1}\tau\sigma$ is an isomorphism from $T$ to $T'$ (so $K \cong K'$) which maps $T_i$ to $T_i'$. As $\alpha$ can be regarded as a semilinear map from $T$ as a vector space over $K$ into $T'$ as a vector space over $K'$, $\alpha$ defines a collineation of $PG(2n+m, s)$ which maps $\mathscr{E}$ onto $\mathscr{E}'$. Hence we have proved the following lemma.

LEMMA 1. *Let $\mathscr{E}_1$, $\mathscr{E}_2$ be two eggs of $PG(2n+m, s)$. There is an isomorphism from $T(\mathscr{E}_1)$ to $T(\mathscr{E}_2)$, which maps the point $(\infty)$ into the point $(\infty)$ if and only if there is a collineation $\alpha$ of $PG(2n+m, s)$ such that $\mathscr{E}_1\alpha = \mathscr{E}_2$.* ∎

If $\mathscr{E}$ is either an oval of $PG(2, q)$ or an ovoid of $PG(3, q)$, then $\mathscr{E}$ is an egg. If this is the case, we will use the standard notation $T_2(\mathscr{E})$ when $\mathscr{E}$ is an oval, and $T_3(\mathscr{E})$ when $\mathscr{E}$ is an ovoid.

Let $\mathscr{E}$ be an egg in $PG(4n-1, s)$. By 8.7.2 of [16] the $s^{2n}+1$ tangent spaces to $\mathscr{E}$ form an egg $\mathscr{E}^*$ with the same parameters in the dual space of $PG(4n-1, s)$. The translation generalized quadrangle $T(\mathscr{E}^*)$ is called the *translation dual* of $T(\mathscr{E})$.

If $\mathscr{F}$ is a semifield flock, the associated generalized quadrangle $Q(\mathscr{F})$ is a translation quadrangle with respect to the line $[A(\infty)]$ ([20]), hence its point-line dual $Q^*(\mathscr{F})$ is isomorphic to the generalized quadrangle $T(\mathscr{E}_{\mathscr{F}})$ for a suitable egg $\mathscr{E}_F$. Recall that if $\mathscr{E}$ is an elliptic quadric of $PG(3, q)$ then so is $\mathscr{E}^*$, hence $T(\mathscr{E})$ and $T(\mathscr{E}^*)$ are isomorphic to the classical generalized quadrangle $Q^-(5, q)$ associated with an elliptic quadric of $PG(5, q)$ and the corresponding flock is linear. S. E. Payne has shown in [13] that if $\mathscr{F}$ is a Kantor flock, then $T(\mathscr{E}_{\mathscr{F}})$ is isomorphic to its translation dual $T(\mathscr{E}^*_{\mathscr{F}})$, whereas if $\mathscr{F}$ is a Ganley flock, then $T(\mathscr{E}^*_{\mathscr{F}})$ is the so-called Roman generalized quadrangle, which is not isomorphic to the point-line dual of a flock quadrangle.

An egg, $\mathscr{E}$, of $PG(4n-1, s)$ is *good* at the element $X$ of $\mathscr{E}$ if any $(3n-1)$-dimensional subspace containing $X$ and at least two other elements of $\mathscr{E}$ contains exactly $s^n+1$ elements of $\mathscr{E}$.

Consider the generalized quadrangle $Q(\mathscr{F})$ arising from a non-linear semifield flock $\mathscr{F}$ and its point-line dual $Q^*(\mathscr{F})$ which is isomorphic to $T(\mathscr{E}_{\mathscr{F}})$ for some egg $\mathscr{E}_F$. As $Q(\mathscr{F})$ is not classical, by 3.3 of [17] the point $(\infty)$ of $Q(\mathscr{F})$ is a line $X$, of type $(b)$, of $T(\mathscr{E}_F)$. By Theorem 6.7 of [23], $\mathscr{E}^*_{\mathscr{F}}$ is good at the tangent space $T_X$ to $\mathscr{E}_{\mathscr{F}}$ at $X$.

THEOREM 1. *If $\mathscr{F}$ is neither linear nor a Kantor semifield flock, then $T(\mathscr{E}^*_{\mathscr{F}})$ is not the point-line dual of a flock generalized quadrangle.*

*Proof.* We recall that a flock generalized quadrangle has a collineation group, of order $q^2$, fixing the special point $(\infty)$, the line $[A(\infty)]$ and acting sharply transitively on the points of $[A(\infty)]$ different from $(\infty)$, because it is an elation generalized quadrangle with base point $(\infty)$. Let $\mathscr{E}_{\mathscr{F}}$ be the egg of $PG(4n-1, s)$, $s^n = q$, associated with the semifield flock generalized quadrangle $Q(\mathscr{F})$. By duality, the line $[A(\infty)]$ of $Q(\mathscr{F})$ corresponds to the point $(\infty)$ of $T(\mathscr{E}_{\mathscr{F}})$ and the point $(\infty)$ of $Q(\mathscr{F})$ corresponds to an element $Y$ of $\mathscr{E}_{\mathscr{F}}$.

By Lemma 1, there is a collineation group $G$ of $PG(4n-1, s)$, of order $q^2$, stabilizing $\mathscr{E}_{\mathscr{F}}$, fixing $Y$, and acting transitively on the elements of $\mathscr{E}_{\mathscr{F}}$ different from $Y$. Therefore, $G$ stabilizes $\mathscr{E}^*_{\mathscr{F}}$, fixes $T_Y$, and acts transitively on the elements of $\mathscr{E}^*_{\mathscr{F}} \setminus \{T_Y\}$.

By way of contradiction, suppose that $T(\mathscr{E}^*_{\mathscr{F}})$ is the point-line dual of a flock generalized quadrangle $\mathscr{S}$.

If $T(\mathscr{E}^*_{\mathscr{F}})$ is not classical, then the point $(\infty)$ of $\mathscr{S}$ corresponds to an element $T_Z$ of $\mathscr{E}^*_{\mathscr{F}}$ ([20] and [23]). If $T(\mathscr{E}^*_{\mathscr{F}})$ is classical, then we can suppose that the point $(\infty)$ of $\mathscr{S}$ corresponds to an element $T_Z$ of $\mathscr{E}^*_{\mathscr{F}}$ because the collineation group of $T(\mathscr{E}^*_{\mathscr{F}})$ is transitive on the points. By Theorem 6.7 of [23], $\mathscr{E}_{\mathscr{F}}$ is a good egg at $Z$. Also, there is a collineation group $H$ of $PG(4n-1, s)$, of order $q^2$, stabilizing $\mathscr{E}^*_{\mathscr{F}}$ and fixing $T_Z$. Hence, $H$ stabilizes $\mathscr{E}_{\mathscr{F}}$ and fixes the element $Z$.

If $Y \neq Z$, the collineation group $\langle H, G \rangle$ of $PG(4n-1, s)$, spanned by $H$ and $G$, acts 2-transitively on $\mathscr{E}_{\mathscr{F}}$. Hence, $\mathscr{E}_{\mathscr{F}}$ is a good egg at all elements. By Section 8.7 of [16], $T(\mathscr{E}_{\mathscr{F}}) \cong Q^-(5, q)$, and $\mathscr{F}$ is a linear flock.

If $Y = Z$, by Corollary 4.2 of [23], $\mathscr{F}$ is a Kantor semifield flock. ∎

## 3. THE SPORADIC SEMIFIELD FLOCK

Recall that an ovoid of $Q(4, q)$ is a set of $q^2+1$ points which has exactly one point in common with each line of $Q(4, q)$. Using the Klein mapping ([6]) from the lineset of $PG(3, q)$ onto the hyperbolic quadric $Q^+(5, q)$ of $PG(5, q)$, to each ovoid $\mathscr{O}$ of $Q(4, q)$ there corresponds a line spread $S(\mathscr{O})$ of $PG(3, q)$, whose lines are totally isotropic with respect to the symplectic polarity of $PG(3, q)$ defined by $Q(4, q)$ via the Klein correspondence. If the translation plane associated with $S(\mathscr{O})$ is a semifield plane, we will say that $\mathscr{O}$ is a *translation ovoid* of $Q(4, q)$.

The known translation ovoids of $Q(4, q)$ are ([18]):

(1) the elliptic quadric $Q^-(3, q)$ intersection of $Q(4, q)$ with a nonsingular hyperplane;

(2) the Kantor ovoid constructed using a Knuth semifield ([10] Sect. 5);

(3) the Thas–Payne ovoid defined in [24] for $q = 3^r$;

(4) the Penttila–Williams ovoid defined in [18] for $q = 3^5 = 243$.

By [12] a translation ovoid[1] defines a semifield flock of the quadratic cone of $PG(3, q)$. By [21] and [22] (Appendix II), an elliptic quadric defines a linear flock, a Kantor ovoid defines a Kantor semifield flock and the Thas–Payne ovoid defines a Ganley flock. In this section we will point out that the Penttila–Williams ovoid defines a new semifield flock which we explicitly describe. The flock is associated with a translation generalized quadrangle and its translation dual which are shown to be new as well.

We first review the construction given in [12].

Denote by $Q(4, q)$ the generalized quadrangle associated with the non-singular quadric of $PG(4, q)$ with equation $x_0x_1 - x_2^2 + x_3x_4 = 0$. If $\pi$ is the plane of $PG(4, q)$ with equations $x_3 = x_4 = 0$, then $C = \pi \cap Q(4, q)$ is a non-singular conic of $\pi$.

If $\Sigma = PG(3, q)$ is the hyperplane of $PG(4, q)$ with equation $x_4 = 0$, then $\pi$ is a plane of $\Sigma$ and we can consider the generalized quadrangle $T_2(C)$, as in the previous section, by viewing the conic $C$ as an egg of $\pi \cong PG(2, q)$.

Fix a point $x$ of $Q(4, q)$ and let $\delta$ be the polarity defined by $Q(4, q)$. If $l$ and $m$ are lines of $Q(4, q)$ and $y$ is a point of $Q(4, q)$, then the map $\theta$, defined by

$$\theta\colon x = (0, 0, 0, 0, 1) \mapsto (\infty),$$

$$\theta\colon l \mapsto l \cap \pi, \qquad \text{for} \quad x \in l \subset x^\delta,$$

$$\theta\colon y \in x^\delta \setminus \{x\} \mapsto y^\delta \cap PG(3, q),$$

$$\theta\colon m \notin x^\delta \mapsto \langle m, (0, 0, 0, 0, 1) \rangle \cap PG(3, q),$$

$$\theta\colon (a, b, c, 1, c^2 - ab) \mapsto (a, b, c, 1, 0),$$

is an isomorphism from $Q(4, q)$ onto $T_2(C)$.

Let $q = s^n$, $V$ be a 3-dimensional vector space over $GF(s^n)$, and $\pi \cong PG(2, s^n) = PG(V, GF(s^n))$. By regarding $V$ as a vector space of dimension $3n$ over $GF(s)$, each point $x$ of $\pi$ defines a $(n-1)$-dimensional subspace $P(x)$ of the projective space $PG(V, GF(s)) = PG(3n-1, s)$, and each line $l$ of $\pi$ defines a $(2n-1)$-dimensional subspace $P(l)$ of $PG(3n-1, s)$. Then $\mathscr{C} = \{P(x) \mid x \in C\}$ is an egg whose tangent spaces are the subspaces $P(l)$ where $l$ is a tangent line to $C$. Moreover, $T(\mathscr{C})$ is a generalized quadrangle isomorphic to $Q(4, s^n)$ ([16] 8.5 and 8.7.1).

From now on, we always suppose that $q = 3^5$, so $s = 3$ and $n = 5$. Fix the cone $K\colon x_0x_1 - x_2^2 = 0 = x_4$. Denote by $[a, b, c, d, 0]$ the plane of $\Sigma$ with equation $ax_0 + bx_1 + cx_2 + dx_3 = 0 = x_4$. Let $\sigma$ be the polarity of $\Sigma$ defined by the map $(a, b, c, d, 0) \mapsto [a, b, c, d, 0]$. Dualizing by $\sigma$, the vertex $(0, 0, 0, 1, 0)$ of $K$ is mapped to the plane $\pi$ and the $q+1$ lines of $K$ are

[1] In [12] translation ovoids appear under the name *semifield* ovoids

mapped to the tangent lines of the conic $C$ of $\pi$ since $4 \equiv 1(mod.\,4)$. The plane $\pi_t$ of the flock $\mathcal{F} = \mathcal{F}(f, g)$ is mapped by $\sigma$ to the point $(t, -f(t), g(t), 1, 0)$ which does not belong to $\pi$, whose equations are $x_3 = x_4 = 0$.

As $q$ is odd, $\mathcal{F} = \mathcal{F}(f, g)$ is a flock if and only if the line of $\Sigma$ joining $(t, -f(t), g(t), 1, 0)$ and $(u, -f(u), g(u), 1, 0)$ intersects $\pi$ in an interior point to $C$. Then $\mathcal{F}(f, g)$ is a semifield flock if and only if $f$ and $g$ are additive and $(t, -f(t), g(t), 0, 0)$ is an interior point to $C$ for all $t$ in $GF(q)$.

If $\mathcal{F} = \mathcal{F}(f, g)$ is a semifield flock, then $f$ and $g$ are $GF(3)$-linear maps of $GF(3^5)$ to themselves, and $L(\mathcal{F}) = \{(t, -f(t), g(t), 0, 0) \mid t \in GF(3^5)\}$ is a 4-subspace of $PG(14, 3)$.

Let $\mathscr{C}$ be the egg of $PG(14, 3)$ constructed above by starting from the conic $C$, and $\hat{\mathscr{C}} = \{P(l) \mid l$ is a tangent line to $C\}$. As the points $(t, -f(t), g(t), 0, 0)$ are internal to $C$ for all $t \in GF(q)$, each element of $\hat{\mathscr{C}}$ is disjoint from $L(\mathcal{F})$.

On the other hand, if $\mathcal{O}$ is a translation ovoid, by the Klein correspondence there is a collineation group $G$ of $Q(4, q)$, of order $q^2$, which fixes a point $x$ of $\mathcal{O}$ and acts transitively on the points distinct from $x$. We notice that if $x = (0, 0, 0, 0, 1)$ then there is an additive map $F$: $GF(q) \times GF(q) \mapsto GF(q)$ such that

$$\mathcal{O} = \{(-t, F(u, t), u, 1, u^2 + tF(t, u)) \mid t, u \in GF(q)\} \cup \{x\}$$

([12] Theorem 5).

If $\theta$ is the isomorphism from $Q(4, q)$ into $T_2(C)$, as defined above, then $(\mathcal{O} \backslash \{(0, 0, 0, 0, 1)\})^\theta = \{(F(u, t), -t, u, 1, 0) \mid t, u \in GF(q)\}$, and $U = \{(F(u, t), -t, u, 0, 0) \mid t,\ u \in GF(q)\}$ is the set of all the points of the intersection of $\pi$ with the lines joining two points of $(\mathcal{O} \backslash \{(0, 0, 0, 0, 1)\})^\theta$.

As the function $F$ is $GF(3)$-linear, $U$ defines a 9-dimensional subspace of $PG(14, 3)$ skew with all elements of $\mathscr{C}$. Denote by $\perp$ the polarity of $PG(14, 3)$ defined by $C$, i.e., $\perp$ is the polarity defined by the quadratic form $\operatorname{trace}(x_0 x_1 - x_2^2) = 0$. Then $U^\perp$ is a subspace of dimension 4 skew with all the elements of $\hat{\mathscr{C}}$. By [12], there is a semifield flock $\mathcal{F}$ of $K$ such that $L(\mathcal{F}) = U^\perp$.

Following the above procedure we are now able to write down the planes of the flock constructed from the Penttila–Williams ovoid.

THEOREM 2. *The flock of the quadratic cone $X_0 X_1 = X_2^2$ in $PG(3, 243)$ associated with the Penttila–Williams ovoid is*

$$\{tX_0 + 2t^9 X_1 + t^{27} X_2 + X_3 = 0 \mid t \in GF(243)\}.$$

*Proof.* We are working in $GF(243) = GF(3^5)$. Fix $Y^5 + 2Y + 1$ as the primitive polynomial defining $GF(3^5)$ over $GF(3)$, so that a primitive element $t$ of $GF(243)$ satisfies $t^5 + 2t + 1 = 0$.

For our computations we explicitly write all elements of $GF(243)$ as a degree four polynomials whose coeffients are in $GF(3)$. Addition is componentwise (with respect to the basis $\{1, t, t^2, t^3, t^4\}$) and multiplication is modulo $t^5+2t+1$.

Let $x_0=\sum u_i t^i$, $x_1=\sum v_i t^i$, $x_2=\sum w_i t^i$ with $u_i$, $v_i$, $w_i \in GF(3)$ and $0 \leqslant i \leqslant 4$.

First, we compute $Q(x_0, x_1, x_2)=\operatorname{trace}(x_0x_1-x_2^2)$:

$$\begin{aligned}\operatorname{trace}(x_0x_1-x_2^2)=\;&2u_0v_0+u_1v_4+u_2v_3+u_3v_2+u_4v_1\\&+u_0v_4+u_1v_3+u_2v_2+u_3v_1+u_4v_0+u_4v_4\\&+w_0^2+w_1w_4+w_2w_3+w_0w_4+w_1w_3+2w_2^2+2w_4^2.\end{aligned}$$

Let $U=\{(0, F(u, t), u, u, -t, 0) \mid t,\ u \in GF(3^5)\}$ be the projection of $\mathcal{O}^\theta$ from $(1, 0, 0, 0, 0, 0)$ onto the plane $\pi$ of $PG(5, 3^5)$ with equations $x_0=x_5=x_2-x_3=0$, which intersects the quadric $Q^+(5, 3^5)$ in the conic

$$\widetilde{\mathscr{C}}: x_1x_4-x_2^2=x_0=x_5=x_2-x_3=0.$$

To simplify notations, we take as local coordinates in $\pi$ $y_0=x_4$, $y_1=x_1$, and $y_2=x_2$ so that the equation for $\widetilde{\mathscr{C}}$ becomes $y_0y_1-y_2^2=0$. Hence the set $\mathscr{L}(\mathscr{F})=\{(t, -f(t), g(t)) \mid t \in GF(3^5)\}$, the polar space of $U$, with respect to the polarity of $PG(14, 3)$ defined by $\operatorname{trace}(y_0y_1-y_2^2)=0$, is in fact, the polar space of the 9-dimensional subspace of $PG(14, 3)$

$$\{(F(u, t), -t, u) \mid u, t \in GF(3^5)\}.$$

Next, we construct $U^\perp$.

The 9-dimensional subspace, $U$, can be associated with the subspace

$$\bar{U}=\{\langle(t^9-u^{81}, -t, u)\rangle \mid t, u \in GF(3^5)\},$$

where $t^9-u^{81}=F(t, u)$. If we let $x_1=\sum v_i t^i=-t$ and $x_2=\sum w_i t^i=u$, then we have $x_0=\sum u_i t^i=-x_1^9-x_2^{81}$ hence we may write the $u_i$'s in terms of $v_i$'s and $w_i$'s obtaining

$$\begin{aligned}u_0&=2v_0+v_1+v_3+v_4+2w_0+w_1+w_2+2w_4,\\u_1&=2v_1+v_2+2w_1+2w_2+2w_3+2w_4,\\u_2&=2v_2+v_3+v_4+2w_1+w_2+2w_4,\\u_3&=v_2+2v_4+w_1+w_2+2w_4,\\u_4&=v_1+v_3+w_1+w_2+w_4.\end{aligned}$$

These equalities will be used later on.

We now would like to determine $\bar{U}^{\perp} = \{\langle \mathbf{y} \rangle \mid f(\mathbf{x}, \mathbf{y}) = 0 \text{ for all } \mathbf{x} \in \bar{U}\}$.

Let $x_0$, $x_1$ and $x_2$ be as above and let $y_0 = \sum k_i t^i$, $y_1 = \sum l_i t^i$ and $y_2 = \sum m_i t^i$ with $k_i, l_i, m_i \in GF(3)$ and $0 \leqslant i \leqslant 4$. Thus, if $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$, then

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) &= Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}) \\ &= 2u_0 l_0 + 2v_0 k_0 + u_1 l_4 + k_1 v_4 + u_2 l_3 + k_2 v_3 \\ &\quad + u_3 l_2 + k_3 v_2 + u_4 l_1 + k_4 v_1 + u_0 l_4 + k_0 v_4 \\ &\quad + u_1 l_3 + k_1 v_3 + u_2 l_2 + k_2 v_2 + u_3 l_1 + k_3 v_1 \\ &\quad + u_4 l_0 + k_4 v_0 + u_4 l_4 + k_4 v_4 \\ &\quad + 2w_0 m_0 + w_1 m_4 + w_4 m_1 + w_3 m_2 + w_2 m_3 \\ &\quad + w_0 m_4 + w_4 m_0 + w_1 m_3 + w_3 m_1 + w_2 m_2 + w_4 m_4. \end{aligned}$$

Since $x_0 = \sum u_i t^i = -x_1^9 - x_2^{81}$ we can eliminate the $u_i$'s and obtain

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) = {} & v_0(2k_0 + k_4 + l_0 + 2l_4) + v_1(k_4 + k_3 + l_1 + 2l_3 + l_4) \\ & + v_2(k_2 + k_3 + 2l_1 + l_4) + v_3(k_1 + k_2 + l_1 + l_2 + l_3 + 2l_4) \\ & + v_4(k_0 + k_1 + k_4 + 2l_0 + 2l_1 + l_3 + l_4) + w_0(2m_0 + m_4 + l_0 + 2l_4) \\ & + w_1(m_3 + m_4 + 2l_1 + l_3 + l_4) + w_2(m_2 + m_3 + 2l_1 + 2l_2 + l_4) \\ & + w_3(m_1 + m_2 + 2l_3 + 2l_4) + w_4(m_0 + m_1 + m_4 + 2l_0 + l_2 + l_3 + 2l_4). \end{aligned}$$

Since $x_1$ and $x_2$ are arbitrary, so are the $v_i$'s and $w_i$'s, hence for $f(\mathbf{x}, \mathbf{y}) = 0$ we require the 10 bracketed terms to be independently zero. Solving in terms of $k_i$ yields

$$\begin{aligned} l_0 &= k_0 + 2k_1 + 2k_3 + 2k_4, \\ l_1 &= k_1 + 2k_2, \\ l_2 &= k_2 + 2k_3 + 2k_4, \\ l_3 &= 2k_2 + k_4, \\ l_4 &= 2k_1 + 2k_3, \\ m_0 &= k_0 + 2k_1 + k_2 + 2k_3 + k_4, \\ m_1 &= k_2 + k_3 + 2k_4, \\ m_2 &= 2k_1 + k_2 + k_3 + 2k_4, \\ m_3 &= 2k_2 + 2k_3, \\ m_4 &= 2k_1 + k_2 + 2k_3 + 2k_4. \end{aligned}$$

Thus given the set of points $\bar{U}^{\perp} = \{(y_0, y_1, y_2)\}$ we have shown that $y_1 = \sum l_i t^i$ and $y_2 = \sum m_i t^i$ are functions of $y_0 = \sum k_i t^i$ hence we Lagrange interpolate to find that

$$\bar{U}^{\perp} = \{\langle (t, 2t^9, t^{27}) \rangle \mid t \in GF(3^5)\} = \{(t, -f(t), g(t)) \mid t \in GF(3^5)\}.$$

Therefore we have obtained the flock

$$(a_i, b_i, c_i) = (t, -f(t), g(t)) = (t, 2t^9, t^{27}), \qquad \text{for} \quad t \in GF(3^5)$$

where the planes of the flock of the quadratic cone $X_0 X_1 - X_2^2 = 0$ of $PG(3, 3^5)$ are $a_i X_0 + b_i X_1 + c_i X_2 + X_3 = 0$. ∎

We now show that the above flock, as well as the associated generalized quadrangle and its translation dual, are new.

THEOREM 3. *If $\mathscr{F}$ is the semifield flock associated with the Penttila–Williams ovoid, then $\mathscr{F}$ is a new semifield flock, which is sporadic, and $Q(\mathscr{F})$ is a new generalized quadrangle.*

*Let $\mathscr{E}_{\mathscr{F}}$ be the egg of $PG(19, 3)$ associated with $\mathscr{F}$. Then the translation dual, $T(\mathscr{E}^*_{\mathscr{F}})$, is a new generalized quadrangle.*

*Proof.* Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two translation ovoids of $Q(4, q)$ and $\mathscr{F}_i$ be the semifield flock associated with $\mathcal{O}_i$ for $i = 1, 2$. By [12] Theorem 3, the ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$ are isomorphic if and only if the flocks are isomorphic.

As the Penttila–Williams ovoid is not isomorphic to any of the other known examples ([18] Theorem 5.3), the semifield flock $\mathscr{F}$ is not isomorphic to either a linear flock, or a Kantor semifield flock, or a Ganley flock. By [2], $\mathscr{F}$ is a sporadic semifield flock.

Now suppose that $Q(\mathscr{F})$ is isomorphic to $Q(\mathscr{F}')$ where $\mathscr{F}'$ is any of linear, Kantor semifield or Ganley flock. Recall that a flock and its derivations give the same generalized quadrangle [17] and that the derived flocks of a semifield flock are pairwise isomorphic ([1] Corollary 1). As $\mathscr{F}$ is not isomorphic to $\mathscr{F}'$, then $\mathscr{F}'$ is isomorphic to each of the derived flocks of $\mathscr{F}$. Fix two of the flocks derived from $\mathscr{F}$, say $\mathscr{F}_0$ and $\mathscr{F}_1$. As $\mathscr{F}_0$ is isomorphic to $\mathscr{F}'$, it is semifield, then the flocks derived from $\mathscr{F}_0$, $\mathscr{F}$ and $\mathscr{F}_1$ included, are pairwise isomorphic so $\mathscr{F}$ is isomorphic to $\mathscr{F}_1$, which is isomorphic to $\mathscr{F}'$, a contradiction.

By Theorem 1, $T(\mathscr{E}^*_{\mathscr{F}})$ is not the point-line dual of a flock generalized quadrangle. We know exactly one translation generalized quadrangle which is not the point-line dual of a flock generalized quadrangle: the translation dual of the egg $\mathscr{E}_{\mathscr{G}}$ where $\mathscr{G}$ is a Ganley flock. (This generalized quadrangle is usually referred to as the *Roman* generalized quadrangle [14, 20].)

If there is an isomorphism from $T(\mathscr{E}^*_{\mathscr{F}})$ onto $T(\mathscr{E}^*_{\mathscr{G}})$, the point $(\infty)$ of $T(\mathscr{E}^*_{\mathscr{F}})$ is mapped to a point $x$ of $T(\mathscr{E}^*_{\mathscr{G}})$. Then $T(\mathscr{E}^*_{\mathscr{G}})$ is a translation generalized quadrangle with base point $x$. We know by the construction of $T(\mathscr{E}^*_{\mathscr{G}})$ as a group coset geometry ([16] 8.2) that the translation group, with base point $x$, of $T(\mathscr{E}_{\mathscr{G}})$ fixes all lines incident with $x$ and is transitive on all the points not collinear with $x$, and, for each line $L$ incident with $x$, acts transitively on the points of $L$ distinct from $x$. By [11] the automorphism group $G$ of $T(\mathscr{E}^*_{\mathscr{G}})$ satisfies one of the following conditions, either

(a) $G$ fixes the point $(\infty)$, or

(b) $G$ fixes a line $L$ incident with $(\infty)$ but no point on it, and $G$ acts 2-transitively on the points of $L$, or

(c) $G$ is transitive on the points of $T(\mathscr{E}^*_{\mathscr{G}})$.

If (a) holds, then $x = (\infty)$ since otherwise the translation group with base point $x$ does not fix $(\infty)$. In the second case, $x$ must be incident with $L$, and there is an automorphism which maps $x$ to $(\infty)$. In (c) there is an automorphism of $T(\mathscr{E}^*_{\mathscr{G}})$ which maps $x$ to $(\infty)$. Therefore, we can always suppose that the given automorphism maps $(\infty)$ to $(\infty)$. By Lemma 1, there is a collineation $\alpha$ of $PG(2n+m, s)$ such that $\mathscr{E}^*_{\mathscr{F}}\alpha = \mathscr{E}^*_{\mathscr{G}}$. By duality, there is a collineation $\beta$ of $PG(2n+m, s)$ such that $\mathscr{E}_{\mathscr{F}}\beta = \mathscr{E}_{\mathscr{G}}$. Then $T(\mathscr{E}_{\mathscr{F}})$ and $T(\mathscr{E}_{\mathscr{G}})$ are isomorphic. Recall that $T(\mathscr{E}_F)$ is isomorphic to the point-line dual of $Q(\mathscr{F})$, similarly for $T(\mathscr{E}_G)$ and $Q(\mathscr{G})$.

As $Q(\mathscr{F})$ and $Q(\mathscr{G})$ are not isomorphic, $T(\mathscr{E}^*_{\mathscr{F}})$ is a new generalized quadrangle. ∎

## 4. CONCLUDING REMARKS

As the generalized quadrangle associated with the new flock is not any of the previously known examples, the flocks derived from this sporadic one, which are pairwise isomorphic, are new ([17]), they are not semifield ([8]), and, as far as we know, the corresponding translation planes, which are isomorphic, are new as well.

The line spread of $PG(3, 3^5)$ associated with the sporadic flock defines a semifield plane of order $3^{10}$ which is new by [5]. This plane is coordinatized by a new semifield $\mathbf{Q}$ of order $3^{10}$ with kernel $K = GF(3^5)$ such that $ba = ab$ for all $a \in \mathbf{Q}$ and $b \in K$. To our knowledge, this is the first example of such a semifield not belonging to an infinite family.

A translation ovoid $\mathcal{O}$ defines a semifield plane as in Section 3. This plane is a flock plane if and only if $\mathcal{O}$ is the union of $q$ conics with a common point ([5]). By construction, this is equivalent to saying the planes of the flock have a common point (recall that $\mathcal{O}$ is contained in a 4-dimensional space). By [19], the flock is linear or Kantor. Therefore, **Q** is not isotopic to the semifield of order $3^{10}$ associated with the Penttila–Williams ovoid via the Klein quadric and constructed in Section 6 of [18].

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